Math 2J
Lecture 16 - 11/02/12

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Markov Chain Recap

- The population of a town is 100000. Each person is either independent, democrat, or republican. In any given year, each person can choose to keep the same affiliation or change. The chances of doing so do not change from year to year.

Representation

$$\vec{p^{t+1}} = T\vec{p^{t}}$$

I D R

$$T = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

D→R

- "T" is a stochastic transition matrix.
- "$a_{i,j}$" represents the chance of moving from state "j" to state "i"

Representation

- "p" is a vector representing the state of the population / system at a particular time

Population Vector

$$\vec{p} = \begin{bmatrix} 10000 \\ 45000 \\ 45000 \end{bmatrix}$$

Relative Fraction

$$\vec{p} = \begin{bmatrix} .1 \\ .45 \\ .45 \end{bmatrix}$$

Probability Vector

Stochastic Matrix / Probability Vector

- Columns of each sum to 1!
- Product of the two is another probability vector!
- The largest eigenvalue of a stochastic matrix is 1!

Markov Chain Example

$$T = \begin{bmatrix} .6 & .1 & .1 \\ .3 & .8 & .1 \\ .1 & .1 & .7 \end{bmatrix}$$

- 80% of democrats stay democrat and 30% of independents become democrat!
  - NOTE : Completely made up numbers!

Eigenvalues

$$\lambda_1 = 1 \qquad \lambda_2 = .6 \qquad \lambda_3 = .5$$

$$\vec{v}_1 = \begin{bmatrix} .2 \\ .55 \\ .25 \end{bmatrix}$$

- All eigenvalues are distinct and det(T)=1*.6*.5=.3.
- So T is invertible, its eigenvectors are linearly independent, and T is diagonalizable.

Convergence

$$\vec{p^0}, \vec{p^1}, \vec{p^2}, \ldots \to \vec{v_1}$$

$$\vec{p^0} = \begin{bmatrix} .6 \\ .1 \\ .3 \end{bmatrix}$$

$$\vec{p^{10}} = T^{10}\vec{p^0} = \begin{bmatrix} .2004 \\ .5493 \\ .2503 \end{bmatrix}$$

$$\vec{p^5} = T^5\vec{p^0} = \begin{bmatrix} .2125 \\ .5336 \\ .2539 \end{bmatrix}$$

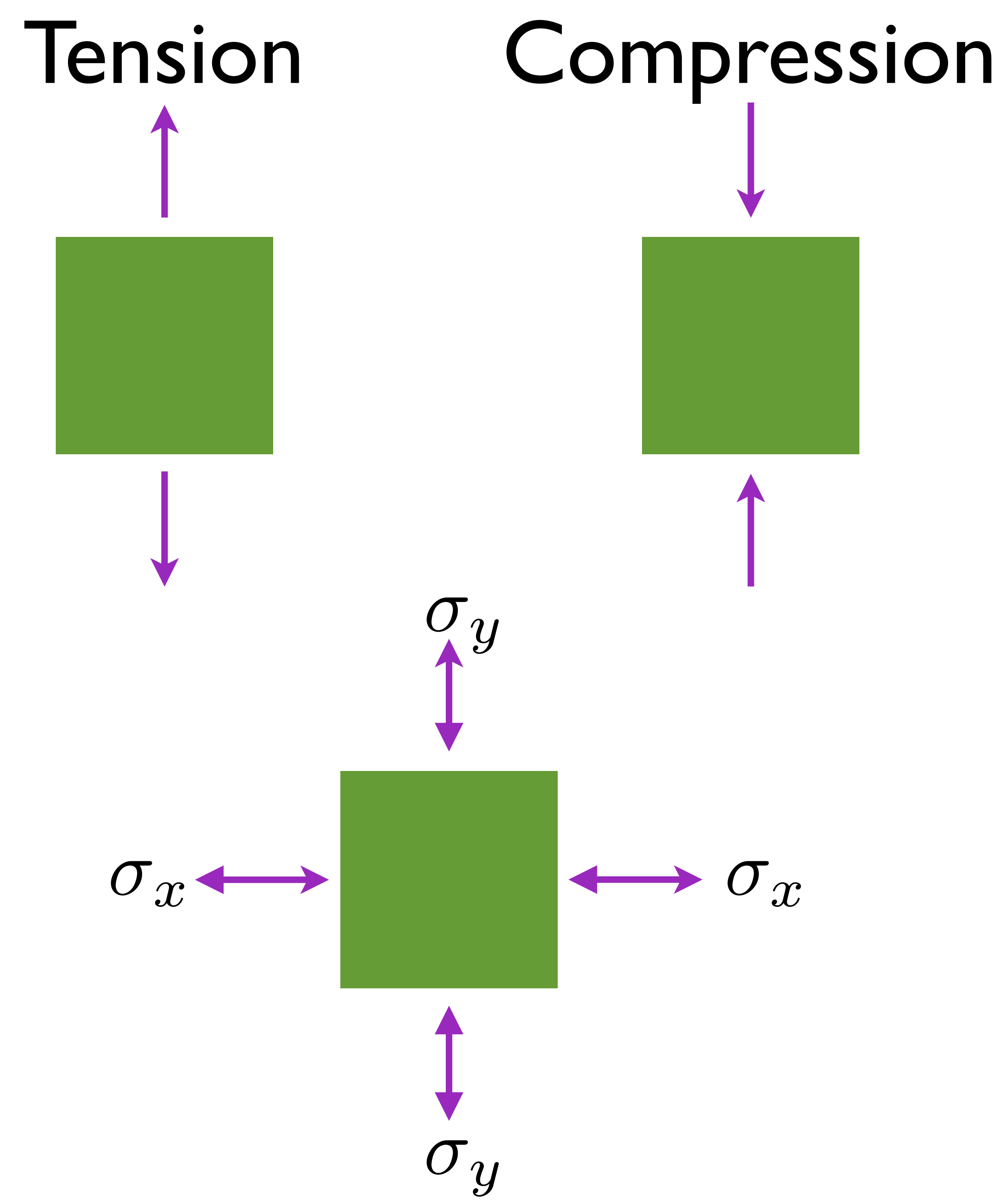
- This clearly converges, and in fact it does so for any initial value (you would not be expected to do this by hand).

Mechanical Force / Stress

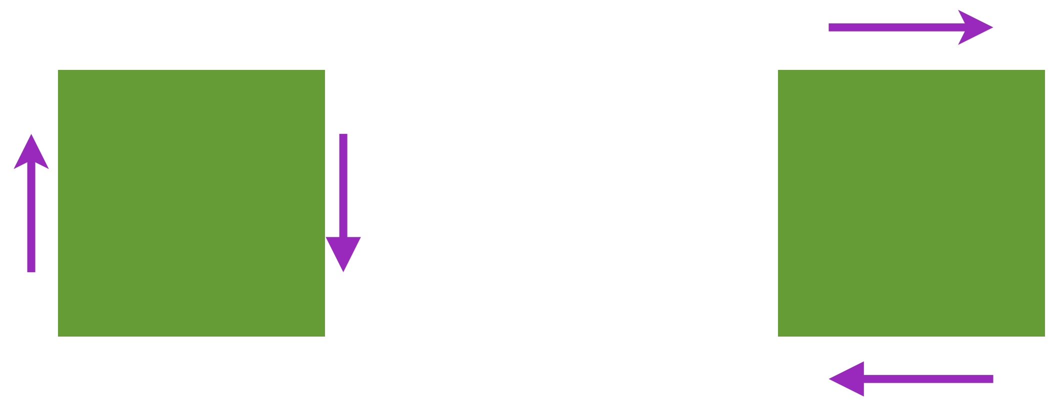
Stress

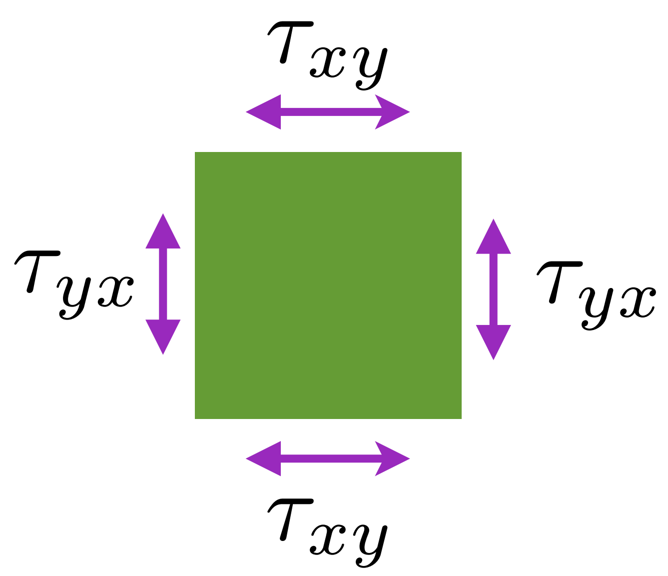
- Measure of the mechanical forces acting on an object.

Normal Stress

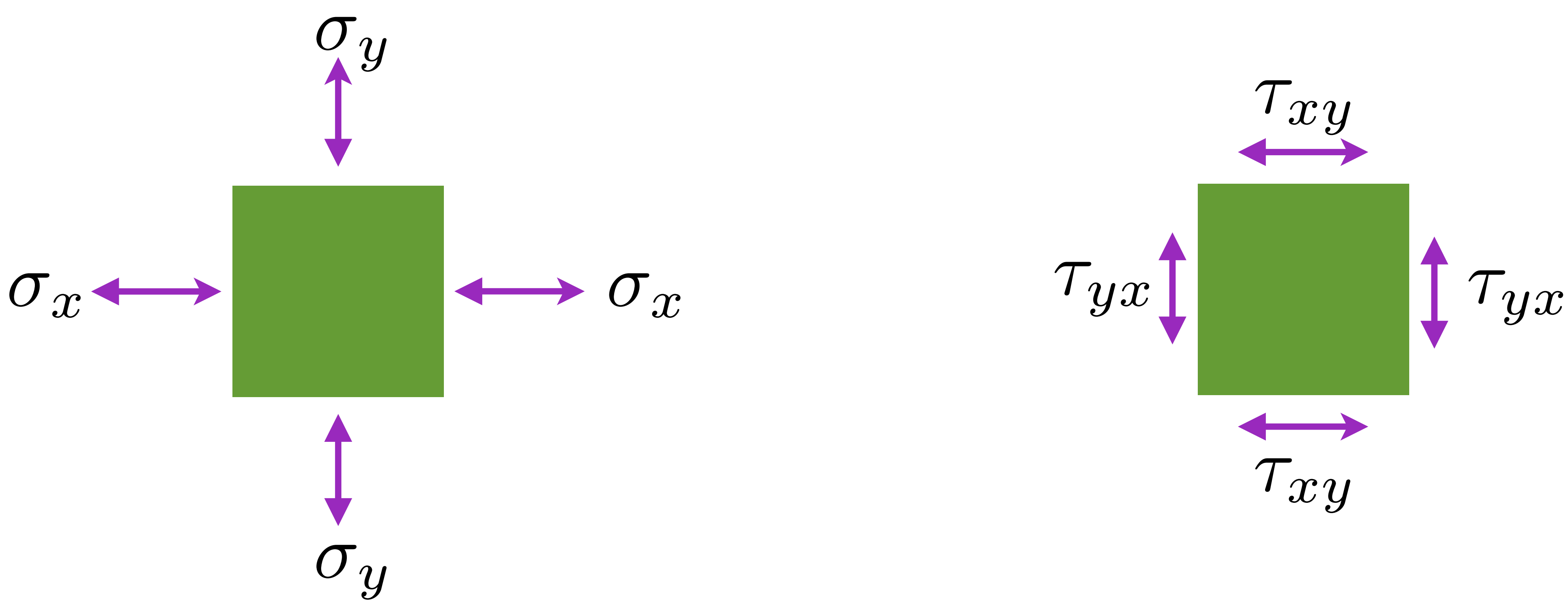


Shear Stress





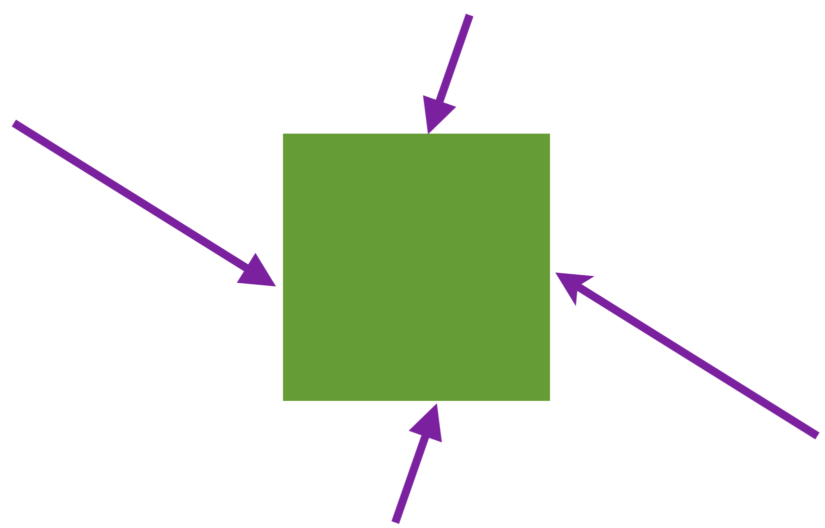
Stress Matrix



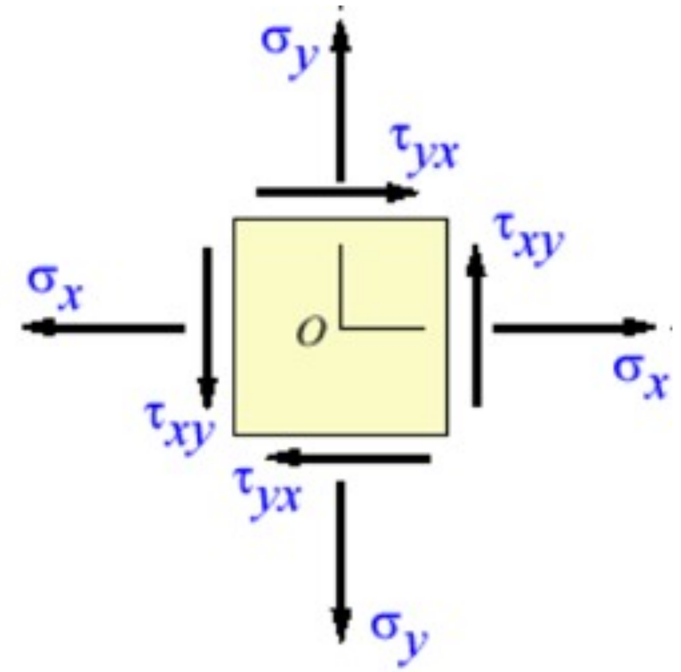
$$S = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

Example

Applied Forces



Resulting Stresses



Stress Matrix

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

Diagonalization

- Find eigenvalues and vectors of "S"

$$\begin{array}{cc} \sigma_1 & \sigma_2 \\ \downarrow & \downarrow \\ \vec{v}_1 & \vec{v}_2 \end{array}$$

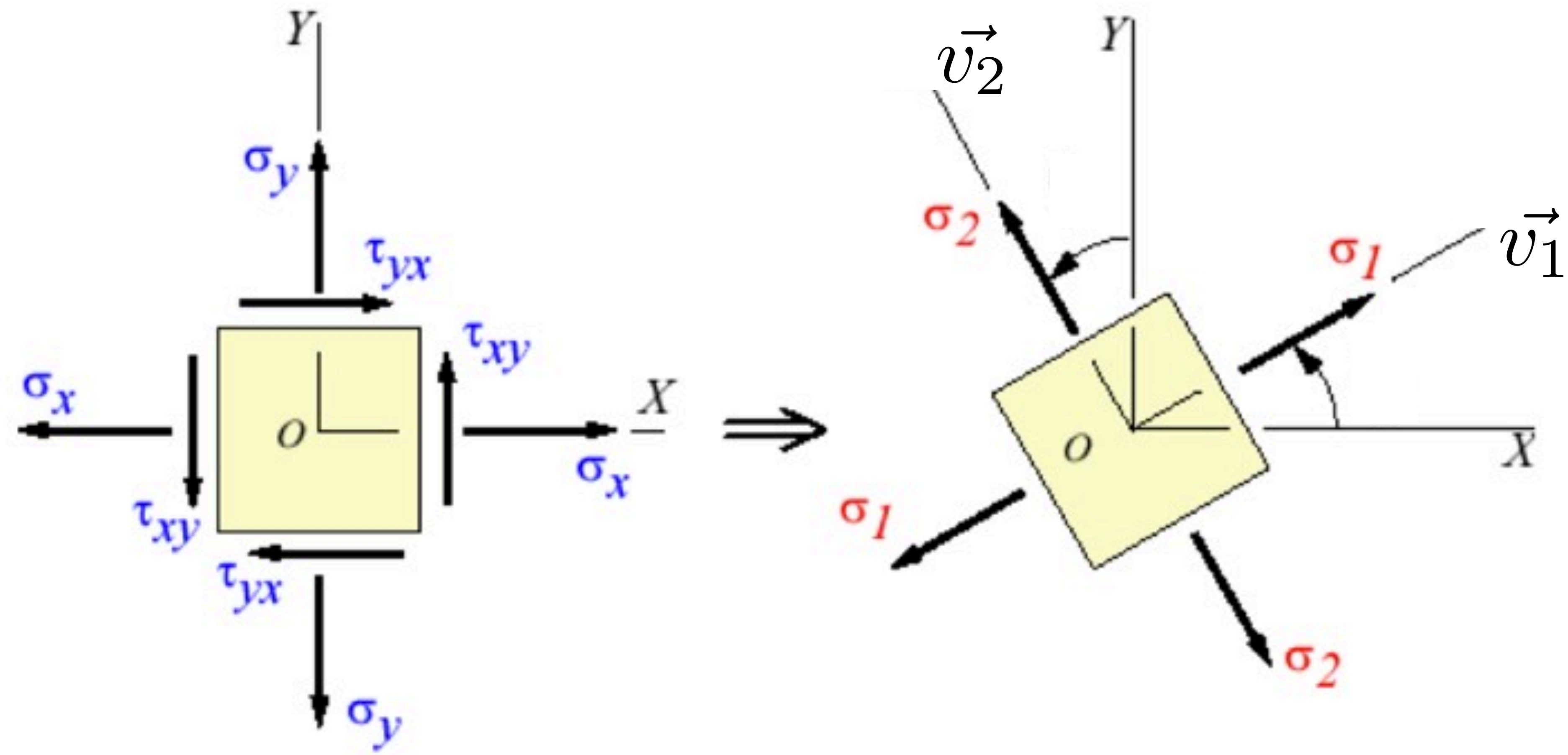
- The eigenvalues will usually be distinct, therefore, the vectors will be linearly independent and S is diagonalizable.

$$V^{-1}SV = D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

Eigenvectors of Stress Matrix

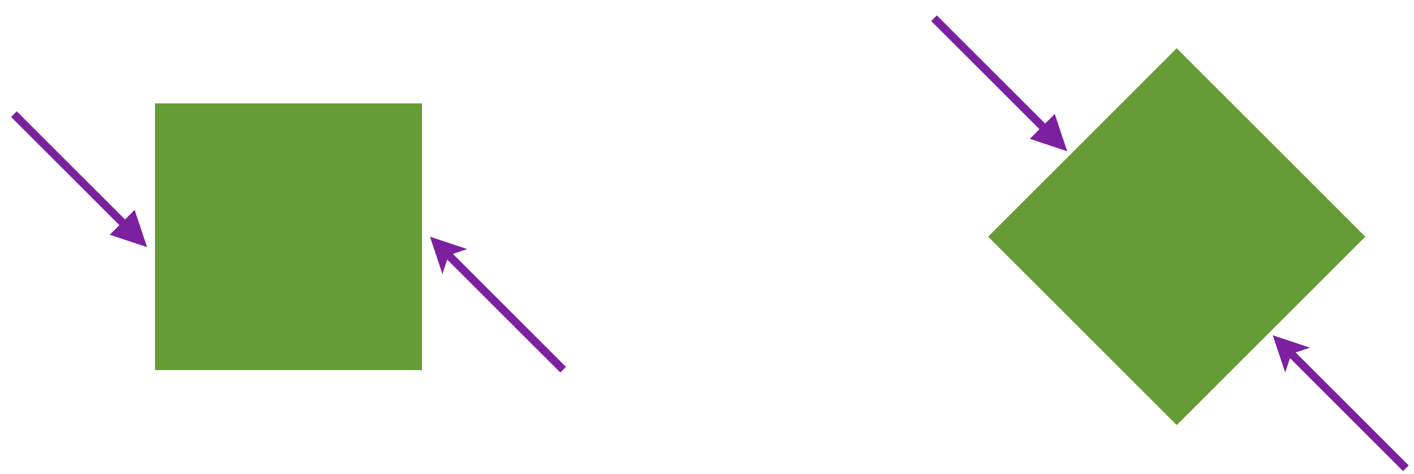
- The eigenvectors are independent and perpendicular.
- They form a new coordinate system for the (x,y) plane.
- In this coordinate system, there is no shear stress.

Coordinate Rotation



Frame of Reference

- The type of stresses you see depends on your frame of reference.

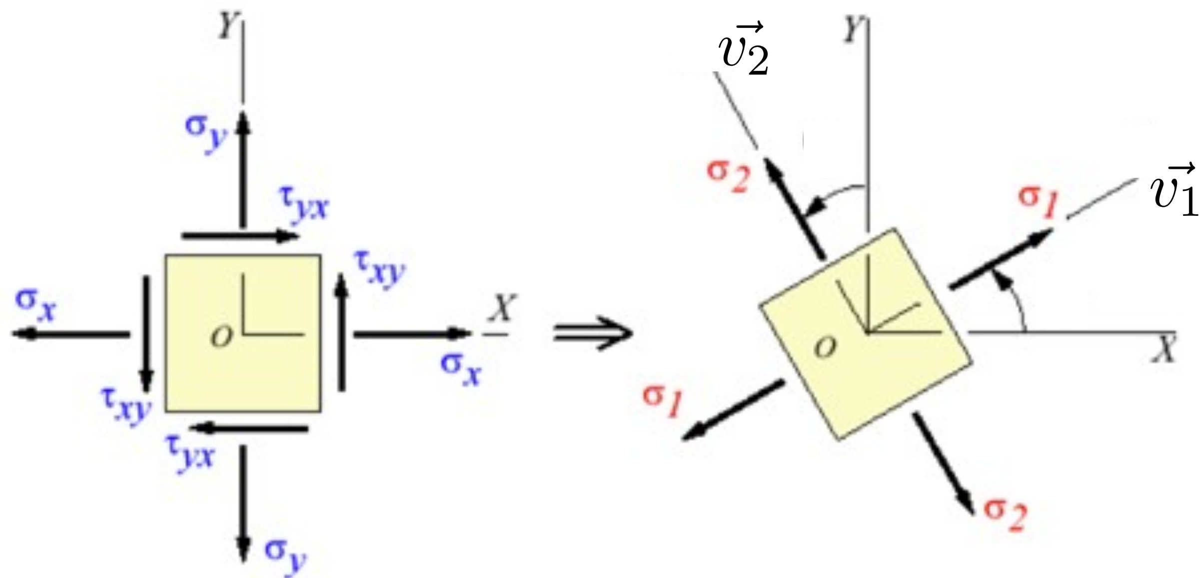


Normal and Shear Stress

Normal Stress Only

- Same forces, different stress profile.

Recap



- Forces determine "S".
- The eigenvectors tell you which frame of reference has no shear stress.
- The eigenvalues tell you what the resulting normal stresses are in that frame of reference.

The Point

- There are many many applications of eigenvalue / vector analysis.
    - Statistics - Principle Component Analysis
    - Computer Science - Google
    - Engineering - Stress analysis
    - and many more.

Exam Topics

- Systems of equations
  - Unique solution, infinately many solutions, no solution.
  - Solving - Gaussian Elimination
  - Solutions with free variables.

Exam Topics

- Matrix Algebra
  - Multiplication, addition, transpose.
  - Inverse
  - Solving a system or matrix equation using the inverse.
  - Determinant.

Exam Topics

- Eigenvalue / vectors.
  - Finding them
  - Their properties
  - Diagonilization
  - Relation to determinant and invertibility.

Exam topics - Applications

- **Markov Chain**
  - Formulating a problem.
  - Finding the future state of a system.
  - Finding the steady state of a system.
  - Stochastic matrices and probability vectors.
- Google page rank
  - Formulating a page rank matrix.
  - It’s properties (ie stochasticity) and eigenvalues.

Not on Exam

- Principle component analysis.
- Stress analysis.
- Any complex fraction or decimal arithmetic that would require a calculator (no calculator allowed).
  - If you're getting bad fractions, or three digit decimals, you're either doing something wrong or I made a mistake, probably the former.